

# Experimental quantum tossing of a single coin

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**Abstract.** The cryptographic protocol of coin tossing consists of two parties, Alice and Bob, that do not trust each other, but want to generate a random bit. If the parties use a classical communication channel and have unlimited computational resources, one of them can always cheat perfectly. Here we analyze in detail how the performance of a quantum coin tossing experiment should be compared to classical protocols, taking into account the inevitable experimental imperfections. We then report an all-optical fiber experiment in which a single coin is tossed whose randomness is higher than achievable by any classical protocol and present some easily realisable cheating strategies by Alice and Bob.

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## 1. Introduction

The cryptographic protocol of coin tossing introduced by Blum [1] consists of two parties, Alice and Bob, that do not trust each other, but want to generate a random bit. If the parties use a classical communication channel and have unlimited computational resources, one of them can always cheat perfectly. But what if they use a quantum communication channel? Because of its conceptual importance and potential applications, quantum coin tossing was already envisaged by Bennett and Brassard in their seminal paper on quantum cryptography [2]. Later works showed that perfect quantum coin tossing is impossible [4, 5, 6], but that imperfect protocols exist [7, 5, 8, 9, 10, 11] that perform better than any classical protocol.

Work on quantum coin tossing distinguishes between “weak coin tossing” and “strong coin tossing”. In weak coin tossing Alice and Bob have antagonistic goals: Alice wants the coin to be heads, say, whereas Bob wants the coin to come out tails. Good quantum protocols for weak coin tossing exist, although they seem very difficult to implement[11]. In strong coin tossing Alice and Bob both want the coin to be perfectly random. Quantum protocols that perform better at strong coin tossing than any classical protocol exist[9, 10] and come close to the known upper bound (for the original unpublished proof of the upperbound, see[6]; published proofs can be found in [12, 13]).

Quantum coin tossing itself is just one example of several interesting tasks that two parties which do not trust each other can achieve if they share a quantum communication channel, but cannot achieve if they use a classical communication channel. Other examples include multiparty coin tossing[12] and weak forms of string commitment[14, 15]. The no go theorems mentioned above [4, 5, 6] rule out most other applications, except if one adds additional assumptions such as bounding the size of quantum memories[16].

Recently two works [17, 18] have experimentally studied optical implementations of quantum coin tossing. However the experiment of ref. [17] suffered from important photon loss which made it difficult to assess how the experiment worked when tossing a single coin. This was circumvented, as in [18], by addressing string flipping, *i.e.* the problem where the parties try to toss a string of coins rather than a single one. These works were however carried out without realizing that good classical protocols exist for string flipping, see e.g. [19] for a presentation of such protocols.

In the present work we go back to the conceptually simpler problem of tossing a single coin, and report an experiment in which a single coin is tossed whose randomness is higher than achievable by any classical protocol. We begin by discussing in detail how the results of such a coin tossing experiment should be compared with classical protocols in view of the inevitable imperfections that will occur in any experimental realisation. Coin tossing in the presence of noise was already studied in [20], but with the emphasis on applications to string flipping, whereas here we are concerned with tossing a single coin. We then present the experimental implementation, which follows

closely the earlier work of [18], and present some easily realisable cheating strategies by Alice and Bob.

## 2. Formulation of the problem

A protocol for coin tossing consists in a series of rounds of (classical or quantum) communication at the end of which the parties decide on an outcome. The outcome can be either a decision that the coin has the value  $c = 0$  or  $c = 1$ , or it can be that the protocol aborts, in which case we say that  $c = \perp$ . Note that because the rounds of (quantum or classical) communication are sequential, it is logically possible for Alice to choose one output  $x$ , and for Bob to choose another output  $y$ . For the sake of generality it is convenient to take this into account and to denote by

$$\begin{aligned} p_{xy} = & \text{Probability that in an honest execution of the} \\ & \text{protocol Alice outputs } x \text{ and Bob outputs } y, \\ & \text{where } x, y \in \{0, 1, \perp\}. \end{aligned}$$

We will say that a protocol is *correct*, if, when both parties are honest, at the end of the protocol they agree on the outcome, and that the results  $c = 0$  and  $c = 1$  occur with equal probability:  $p_{00} = p_{11} = (1 - p_{\perp\perp})/2$ . This formulation takes into account that because of experimental imperfections, the outcome  $c = \perp$  may occur even when both parties are honest.

The aim of a cheater is to force the outcome of the coin tossing protocol. We denote by

$$\begin{aligned} p_{*y} = & \text{Probability that a dishonest Alice can force an} \\ & \text{honest Bob to output } y \\ p_{x*} = & \text{Probability that a dishonest Bob can force an} \\ & \text{honest Alice to output } x \end{aligned}$$

An alternative notation often used in the literature is the bias  $\epsilon$  which is related to our notation by

$$\begin{aligned} \epsilon_A &= \max_y (p_{*y} - \frac{1}{2}) \\ \epsilon_B &= \max_x (p_{x*} - \frac{1}{2}) \end{aligned} \tag{1}$$

The bound due to Kitaev [6, 12, 13] states that either  $\epsilon_A$  or  $\epsilon_B$  is greater or equal to  $1/\sqrt{2}$ . The best known protocol for strong coin tossing due to Ambainis has  $\epsilon_A = \epsilon_B = 1/4$ . In our experimental implementation, as we will see later, we will be concerned by a protocol which in the terminology of [10] has “ $\rho_0$  and  $\rho_1$  both pure”. For such protocols it is proven in [10] that  $\epsilon_A^2 + \epsilon_B^2 \geq 1/4$ .

In the appendix we prove the following (which generalises a result of [6] when  $p_{\perp\perp} = 0$ ):

*Lemma 1: For any correct classical coin tossing protocol with three outcomes  $0, 1, \perp$  we have:*

$$(1 - p_{0*})(1 - p_{*1}) \leq p_{\perp\perp} , \quad (2)$$

$$(1 - p_{1*})(1 - p_{*0}) \leq p_{\perp\perp} . \quad (3)$$

Note that if  $p_{\perp\perp} = 0$  these inequalities imply that either  $p_{0*} = 1$  or  $p_{*1} = 1$ , and that either  $p_{1*} = 1$  or  $p_{*0} = 1$ , thereby showing that classical coin tossing is impossible. When  $p_{\perp\perp} \neq 0$  a cheater can no longer necessarily force the outcome he wants. In the supplementary material we show that there exist classical protocols that saturate either one of equations (2) or (3), and that there exist classical protocols that come close to saturating both equations (2) and (3).

In view of Lemma 1, it is natural to quantify the quality of quantum coin tossing experiments by the following merit function:

$$\mathcal{M} = \frac{(1 - p_{*0})(1 - p_{1*})}{2} + \frac{(1 - p_{*1})(1 - p_{0*})}{2} - p_{\perp\perp} \quad (4)$$

which has the following properties:

- (i) Positivity of probabilities implies  $-1 \leq \mathcal{M} \leq +1$
- (ii) For any classical protocol we have  $\mathcal{M} \leq 0$
- (iii) An ideal protocol would have  $\mathcal{M} = 1$ .

The interpretation of the merit function is most obvious in the weak coin tossing scheme wherein Alice wins if Bob outputs 1 while Bob wins if Alice outputs 0 because then the term  $(1 - p_{*1})(1 - p_{0*})$  is the product of how often a dishonest Alice cannot force a win times how often a dishonest Bob cannot force a win (and similarly for the term  $(1 - p_{*0})(1 - p_{1*})$ ). The better the protocol, the larger these terms.

As illustration let us compute the value of  $\mathcal{M}$  for different protocols. The bound due to Kitaev states with precision, see [12], that  $p_{*1}p_{1*} \geq 1/2$  and  $p_{*0}p_{0*} \geq 1/2$ . Inserting this into eq . (4) shows that for all quantum protocols,  $\mathcal{M} \leq (1 - 1/\sqrt{2})^2 \simeq 0.086$ . For Ambainis's protocol [9] for instance we have  $\mathcal{M} = 1/16 = 0.0625$ .

### 3. Experimental Implementation

#### 3.1. The Protocol

Our implementation of quantum coin tossing uses the following protocol:

- (i) Alice chooses  $a \in \{0, 1\}$  at random. She prepares state  $\psi_a$ , where the two possible states are non orthogonal:  $|\langle\psi_1|\psi_0\rangle| = \cos\theta > 0$ . She sends  $\psi_a$  to Bob.

The states  $\psi_{0,1}$  will be taken to be coherent states of light of amplitude  $\alpha$  and opposite phase:

$$|\psi_0\rangle = |+\alpha\rangle \quad , \quad |\psi_1\rangle = |-\alpha\rangle \quad (5)$$

which implies that

$$\cos^2 \theta = |\langle \psi_1 | \psi_0 \rangle|^2 = |\langle -\alpha | + \alpha \rangle|^2 = e^{-4\alpha^2}. \quad (6)$$

In the notation of [10] we thus have  $\rho_0 = |\psi_0\rangle\langle\psi_0|$  and  $\rho_1 = |\psi_1\rangle\langle\psi_1|$  both pure. Also note that  $\rho_0 \neq \rho_1$  prevents from cheating strategies based on entanglement [3, 4].

- (ii) Bob chooses  $b \in \{0, 1\}$  at random. He tells the value of  $b$  to Alice.
- (iii) Alice tells Bob the value of  $a$ .
- (iv) Bob carries out a measurement which projects onto  $\psi_a$  or onto the orthogonal space. If he finds that the state is not equal to  $\psi_a$  he aborts, and the outcome of the protocol is  $\perp$ . If he finds that the state is equal to  $\psi_a$  then the outcome of the protocol is  $c = a \oplus b$ .

Bob's measurement is carried out as follows: using a local oscillator (LO), he displaces the quantum state by  $+\alpha$  if  $a = 1$  or by  $-\alpha$  if  $a = 0$ . If Alice is honest this results in the state becoming the vacuum state. To check this Bob then sends the resulting state onto a single photon detector. If the detector clicks then Bob assumes that Alice was cheating and he aborts: the outcome of the protocol is  $\perp$ . If the detector does not click, then Bob assumes that Alice is honest. (Note that Bob's measurement is similar in spirit to the method proposed in [21] for quantum state tomography, but Bob's task is simpler since he only needs to detect if Alice is cheating, and not carry out the full state tomography).

### 3.2. Analysis in the absence of imperfections

We now study how the merit function  $\mathcal{M}$  depends on the details of the experiment. For the sake of comparison we first look at the situation in the absence of imperfections.

First of all, in this case  $p_{\perp\perp} = 0$ .

Second, if Alice is dishonest she will send a fixed state  $|\phi\rangle$  at step 1 and at step 3 she will choose the value of  $a$  which will make her win the protocol, and then she will hope that Bob will not abort. The probability that Bob will abort is given by the overlap of  $|\phi\rangle$  with  $|\psi_0\rangle$  and  $|\psi_1\rangle$ . One easily finds (see [20]) that Alice's optimal choice is  $|\phi\rangle = N(|\psi_0\rangle + |\psi_1\rangle)$  where  $N$  a normalization constant, yielding the optimal values:

$$p_{*0} = p_{*1} = \frac{1}{2} + \frac{|\langle \psi_1 | \psi_0 \rangle|}{2} = \frac{1}{2} + \frac{\cos \theta}{2}. \quad (7)$$

Third, if Bob is dishonest, he will measure the state sent by Alice at step 2 so as to try to find out whether it is  $\psi_0$  or  $\psi_1$ , and he will then choose the value of  $b$  according to the result of his measurement. For the optimal measurement the probability that Bob wins is

$$p_{0*} = p_{1*} = \frac{1}{2} + \frac{\sqrt{1 - |\langle \psi_1 | \psi_0 \rangle|^2}}{2} = \frac{1}{2} + \frac{\sin \theta}{2}. \quad (8)$$

The maximal value of the merit function  $\mathcal{M}_{max} = \frac{(1-1/\sqrt{2})^2}{4} \simeq 0.021$  occurs when  $\cos(\theta) = \sin(\theta) = 1/\sqrt{2}$ , corresponding to  $\alpha^2 = 0.17$ . Note that this is the maximum

value for protocols which in the terminology of Spekkens and Rudolph [10] fall in the category “ $\rho_0$  and  $\rho_1$  both pure”.

### 3.3. Analysis in the presence of imperfections

To obtain estimates on  $p_{*c}$ ,  $p_{c*}$  and  $p_{\perp\perp}$ , and hence to estimate  $\mathcal{M}$ , in the presence of imperfections requires that we make assumptions on how the experiment is carried out.

The parameter,  $p_{\perp\perp}$ , which we also call the Quantum Bit Error Rate (QBER), can easily be measured experimentally by tossing a large number of coins with Alice and Bob both following their honest strategy.

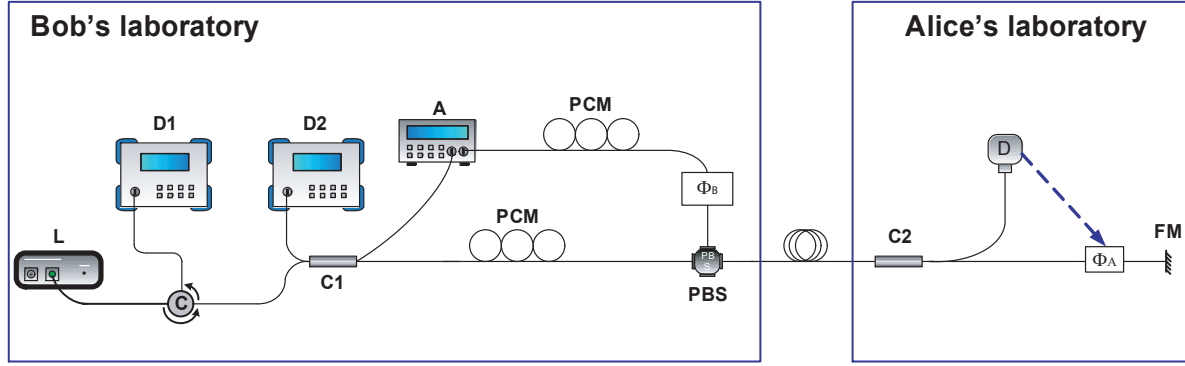
**3.3.1. Bob is dishonest** When Bob is dishonest his cheating strategy is, as before, to estimate before step 2 the state  $|\psi_a\rangle$  prepared by Alice so as to correctly guess the value of  $a$ . How do experimental imperfections, and in particular the limited visibility  $V$  of interferences affect Bob’s success probability  $p_{c*}$ ? To analyse this note that the state Alice sends to Bob is a short laser pulse of known intensity which is then strongly attenuated. Under strong attenuation all quantum states tend towards mixtures of coherent states (see e.g. [18]). Thus we can assume that the states prepared by Alice are coherent states of known intensity  $\alpha^2$ . These coherent states are not precisely known to Alice. However it is not difficult to show that if two coherent states have intensity  $\alpha^2$ , their scalar product is lower bounded by  $|\langle\psi_1|\psi_0\rangle| \geq e^{-2\alpha^2}$ . Bob’s cheating probability can then be bounded, as in equation (8), by the scalar product of the two states prepared by Alice:

$$p_{c*} = \frac{1}{2} + \frac{\sqrt{1 - |\langle\psi_1|\psi_0\rangle|^2}}{2} \leq \frac{1}{2} + \frac{\sqrt{1 - e^{-4\alpha^2}}}{2} . \quad (9)$$

**3.3.2. Alice is dishonest** When Alice is dishonest we suppose that she can prepare an arbitrary state just in front of Bob’s laboratory, and then send it to Bob. How do the imperfections in Bob’s laboratory affect  $p_{*c}$ ? To quantify this Bob could carry out a complete tomography of his measurement apparatus, and based on the results compute what is Alice’s best cheating strategy. Here we will make a simple estimate based on easily accessible parameters.

First of all let us consider the effects of the attenuation  $A_T$  during transmission between Alice and Bob’s laboratories, of the attenuation  $A_B$  in Bob’s apparatus, and of the efficiency  $\eta$  of his detector. We take these parameters into account by analysing a fictitious system in which Bob’s apparatus is replaced by a lossless apparatus, and all the attenuation is under Alice’s control, *i.e.*  $\eta^{fict} = 100\%$ ,  $A_B^{fict} = 1$ , and  $A_T^{fict} = A_T A_B \eta$ . This replacement can only help a cheating Alice. In the fictitious system the state sent by an honest Alice is  $|\pm \alpha_B^{fict}\rangle = |\pm \alpha \sqrt{A_T A_B \eta}\rangle$ .

Second we analyse the effect of finite visibility on the performance of the fictitious system just described. Because of the finite visibility, Bob will not be making a projection onto the state  $|\pm \alpha_B^{fict}\rangle$ , but onto slightly different states. We make the



**Figure 1.** Experimental setup: L, picosecond laser; D1, main photon counter; D2, auxiliary photon counter; C1, 50/50 coupler; A, attenuator;  $\Phi_B$  Bob's phase modulator; PCM, polarization controller; PBS, polarizing beam splitter; C2, 80/20 coupler; PD, photodiode;  $\Phi_A$ , Alice's phase modulator; FM, Faraday mirror.

assumption that Bob's apparatus acts as a passive linear optical system. This implies that the true states onto which Bob projects are slightly modified coherent states  $|\pm \alpha_B^{fict} + \delta_{\pm}\rangle$ . The deviations due to  $\delta_{\pm}$  give rise to the optical contribution to the QBER:

$$QBER_{opt} = (|\delta_+|^2 + |\delta_-|^2) / 2 = q |\alpha_B^{fict}|^2, \quad (10)$$

where  $q$ , the QBER per photon, can be related to the visibility  $V$  of interferences by  $q \simeq (1 - V)/2$ . (Note that in addition to  $QBER_{opt}$ , there is another contribution to the QBER due to the dark counts of the detectors. The total QBER is the sum of these two contributions:  $QBER = QBER_{opt} + QBER_{dk}$ .)

The distance between the two states onto which Bob projects is given by

$$\begin{aligned} & |(+\alpha_B^{fict} + \delta_+) - (-\alpha_B^{fict} + \delta_-)|^2 \\ & \geq 4|\alpha_B^{fict}|^2 - 4|\alpha_B^{fict}||\delta_+ - \delta_-| = 4|\alpha_B^{fict}|^2(1 - 2\sqrt{q}). \end{aligned} \quad (11)$$

Inserting this into equation (7) gives

$$p_{*c} \leq \frac{1}{2} + \frac{1}{2} \exp \left[ -A_B A_T \eta (1 - 2\sqrt{q}) \alpha^2 \right]. \quad (12)$$

Thus the effect of the imperfections is to replace  $\alpha^2$  by an effective attenuated intensity  $A_B A_T \eta (1 - 2\sqrt{q}) \alpha^2$ .

### 3.4. Experimental results

Our experimental setup, depicted in Fig. 1, based on the plug and play system developed for long distance quantum key distribution [22], is very similar to the one described in [18]. It consists of an all-fiber (standard SMF-28) passively balanced interferometer, and is therefore well suited to long distance quantum communication. The protocol begins with Bob producing a short (300 ps) intense laser pulse at  $\lambda = 1.55 \mu m$  (id300



from idQuantique). The pulse is split in two by the coupler C1 with equal reflection and transmission coefficients 50%. The two pulses are delayed one with respect to the other by 134ns. The pulses are then recombined on a polarizing beam splitter (PBS) and sent to Alice. The pulse that propagated along the long arm of the interferometer is strongly attenuated and will play the role of signal. The pulse that propagated along the short arm will play the role of local oscillator (LO). Upon receiving the pulses, Alice splits off part of them using the coupler C2 and sends this to a photodiode that triggers her electronics. At Alice's site the pulses are further attenuated by the different optical elements. They are reflected by the Faraday mirror. And Alice randomly chooses which phase  $\Phi_A = 0, \pi$  to put on the signal pulse using her phase modulator. The signal Alice sends back to Bob is thus the coherent state  $|\pm\alpha\rangle$  with average photon number  $|\alpha|^2 = 0.27$ .

When the pulses come back to Bob's site, they are sent along the short and long arm of the interferometer by the PBS and interfere at coupler C1. In front of the PBS is a delay line belonging to Bob which ensures that after the pulses enters Bob's laboratory he has the time to send to Alice the value of the bit  $b$  and then receive from her the value of  $a$ . In our experiment the fiber pigtailed of the PBS are sufficient to realize the delay. Upon receiving the value of  $a$ , Bob puts the corresponding phase  $\Phi_B = a\pi$  on the LO. This ensures that there should be destructive interference at the output port that goes to the circulator and then to detector D1 (id200 from idQuantique). If detector D1 registers a click, Bob aborts. If it does not click, the outcome of the coin toss is  $c = a \oplus b$ . The other output of coupler C1 is monitored by detector D2, although this is not directly used in the experiment.

There are in fact two security loopholes in this experiment. The first arises because Alice does not know the intensity of the signal pulse she attenuates before sending it back to Bob. Thus in principle Bob could send her a more intense state than expected, which would mean that the scalar product of the states prepared by Alice would be smaller than expected. The second security loophole arises because Bob does not know the intensity of the pulse he uses as LO. Thus in principle Alice could send Bob the vacuum state, both in the signal and LO, and cheat perfectly. Both loopholes could be closed by having Alice (Bob) monitor the intensity of the signal (LO) before she (he) attenuates it. This was not realised in the present setup because the laser pulses used were not intense enough, but would be possible using more intense or longer laser pulses as in [18], or by using an isolator combined with an amplitude modulator as in [24].

*3.4.1. Both parties are honest* As mentioned above, we performed the experiment with  $|\alpha|^2 = 0.27$ . In a typical series 10000 coins were tossed, and we obtained 5066 occurrences of  $c = 1$ , 2 occurrences of  $c = \perp$ , the other outcomes being  $c = 0$  (which is consistent with the statistical uncertainty which should be of order  $\sqrt{5000} = 70$ ). However we insist that the protocol can be used to toss a single coin.

We estimate the merit function as follows. The abort probability is estimated by



tossing a large number ( $1.5 \cdot 10^5$ ) of coins with Alice and Bob both honest:

$$p_{\perp\perp} \simeq 1.40 \pm 0.37 \cdot 10^{-4} . \quad (13)$$

where the error comes from statistical uncertainty.

The transmission losses are assumed to be negligible,  $A_T = 1$ , as both parties are separated by a few meters of optical fiber. Bob's detector D1 has a  $\eta = 10\%$  quantum efficiency. It is gated using a  $2.5 \text{ ns}$  gate leading to a dark count probability of  $4.7 \cdot 10^{-5}$ . The attenuation of the signal in the optical elements of Bob's laboratory has been measured to be  $A_B \simeq -6 \text{ dB}$  (which includes the 3 dB losses at coupler C1 where the signal and the LO interfere). Visibilities, as measured using an intense signal, were at least 99.0% (corresponding to  $q = 5 \cdot 10^{-3}$ ). By inserting these parameters in equations (9) and (12) we obtain upper bounds for  $p_{*c}$  and  $p_{c*}$ :

$$p_{*c} \leq 0.9971 \quad \text{and} \quad p_{c*} \leq 0.906 \quad (14)$$

leading to the lower bound for the merit function:

$$\mathcal{M} \geq 1.33 \cdot 10^{-4} . \quad (15)$$

This bound may seem very small. Its value is roughly explained by noting that the maximal value in the absence of imperfections is  $\mathcal{M}_{max} = 0.021$ . The main source of imperfections are the efficiency of the detectors (10 dB) and the losses in Bob's apparatus (6 dB). Thus we should reduce the attainable value of  $\mathcal{M}$  by a factor 40, yielding approximately equation (15). This argument shows that the simplest way to improve the experiment would be to use a more efficient detector. It also shows that the value of  $\mathcal{M}$  is rather robust against small variations of the experimental parameters. We have computed that we could keep  $\mathcal{M}$  positive while increasing losses between Alice and Bob to  $A_T \simeq 4.4 \text{ dB}$  (more than 20 km of SMF-28 fiber), all other parameters being kept constant.

*3.4.2. Bob is dishonest* In order to cheat Bob must estimate the state  $|\psi_a\rangle$  prepared by Alice so as to correctly guess the value of  $a$  before sending the value of  $b$ . We implemented a simple cheating strategy in which Bob always applies  $\Phi_B = 0$  on the LO. If detector D1 clicks Bob assumes that Alice chose  $a = 1$ , whereas if D1 does not click he assumes  $a = 0$ . Implementing this strategy yielded the value  $p_{1*} = 0.505$ . This very low value is due to the small values of  $\eta$  and  $A_B$ . Note that a much better cheating strategy, but which was impossible to implement in our laboratory, would be for Bob to carry out a homodyne measurement and measure the quadrature that gives him the best estimate of  $a$ .

*3.4.3. Alice is dishonest* As discussed above, when Alice is dishonest her best strategy is to send a fixed state  $|\phi\rangle = N(|+\alpha\rangle + |-\alpha\rangle)$  to Bob. After receiving  $b$  she then sends the value of  $a$  that makes her win the coin toss and hopes that Bob will not abort. In practice we implemented a strategy where Alice always sends  $|+\alpha\rangle$ . Even though this strategy is very basic, it leads to  $p_{*c} = 0.9956$ , which is very close to the theoretical maximum equation (14).

#### 4. Conclusion

In conclusion we have studied in detail how the performance of quantum coin tossing protocols in the presence of imperfections should be compared to classical protocols. We then reported on a fiber optics experimental realisation of a quantum coin tossing protocol. Our analysis shows that in this realisation the maximum success cheating probabilities for Alice and Bob are respectively 0.9971 and 0.906 when experimental imperfections are taken into account, which is still better than achievable by any classical protocol. We implemented this protocol using an all-optical fiber scheme and tossed a coin whose randomness is higher than achievable by any classical protocol. Finally we implemented simple realisable cheating strategies for both Alice and Bob.

After the present work was completed, we learned of a recent proposal specially designed for carrying out quantum coin tossing in the presence of losses[23]. Obviously taking into account losses, in particular those that occur in Bob's apparatus, was an important consideration when choosing and analysing the protocol reported here. The protocol reported in [23] seems more tolerant to loss than ours. Once the effect of other imperfections (such as finite visibility of interference fringes) are taken into account, it could be compared to ours using the merit function  $\mathcal{M}$  introduced above.

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#### Appendix

Here we provide bounds on the performance of classical coin tossing protocols when there is some probability that the protocol aborts when both parties are honest. We also show that there exist classical protocols that attain these bounds. We use the notation and terminology introduced in the main text. The idea of the following result is to analyse the performance of a classical protocol with 3 outcomes (*i.e.* a classical protocol in which the parties try to toss a trit.).

*Lemma 1:* For any correct classical coin tossing protocol with three outcomes 0, 1,  $\perp$  we have:

$$(1 - p_{0*})(1 - p_{*1}) \leq p_{\perp\perp} , \quad (\text{A.1})$$

$$(1 - p_{1*})(1 - p_{*0}) \leq p_{\perp\perp} . \quad (\text{A.2})$$

*Proof of Lemma 1.* We need to introduce some notation.

The protocol consists of  $K$  rounds of communication, labeled  $j = 1, \dots, K$ .

Denote by  $u_j$  the possible states of the protocol at round  $j$ .

Denote by  $w(u_j)$  the probability of reaching state  $u_j$  at round  $j$  in an honest execution of the protocol.

Denote by  $w(u_{j+1}|u_j)$  the probability that in an honest execution, the protocol will be in state  $u_{j+1}$  at round  $j + 1$  if it is in state  $u_j$  at round  $j$ .

Denote by  $p_{*y}(u_j)$  the maximum probability that if Alice is dishonest and Bob is honest, then Alice can force Bob to output  $y$  at the end of the protocol if the state at round  $j$  is  $u_j$ .

Denote by  $p_{x*}(u_j)$  the maximum probability that if Bob is dishonest and Alice is honest, then Bob can force Alice to output  $x$  at the end of the protocol if the state at round  $j$  is  $u_j$ .

Introduce the quantity  $T_j$  defined by

$$T_j(x, y) = \sum_{u_j} w(u_j)(1 - p_{x*}(u_j))(1 - p_{*y}(u_j))$$

Note that if we take  $x = 0$  and  $y = 1$  the initial value ( $j = 1$ ) of  $T$  is  $T_1(0, 1) = (1 - p_{0*})(1 - p_{*1})$ , ie. the left hand side of eq. (A.1).

Note also that at round  $K$ , when the protocol has ended,  $T_K$  is equal to the sum over the final states of the protocol in an honest execution of the product of the probabilities that the output of Alice is not  $x$  and that the output of Bob is not  $y$ . Thus if we take  $x = 0$  and  $y = 1$ , then  $T_K(0, 1) = p_{\perp\perp}$ , ie. the right hand side of eq. (A.1).

To complete the proof we show that  $T$  is an increasing function of  $j$ , ie.  $T_{j+1} \geq T_j$ . To this end suppose that at round  $j$  Bob will send some communication to Alice.

Then Alice cannot influence what will happen at round  $j$ , hence we have:  $p_{*y}(u_j) = \sum_{u_{j+1}} w(u_{j+1}|u_j)p_{*y}(u_{j+1})$ .

Furthermore we have the trivial identity  $w(u_{j+1}) = \sum_{u_j} w(u_{j+1}|u_j)w(u_j)$ .

Finally we note that since it is Bob's turn to talk at round  $j$ , we have  $1 - p_{x*}(u_j) \leq 1 - p_{x*}(u_{j+1})$  where  $u_{j+1}$  is any state at round  $j + 1$  that can be obtained from state  $u_j$  at round  $j$  in an honest execution.

Inserting these identities into the definition of  $T_j$ , we obtained the desired inequality  $T_{j+1} \geq T_j$ .

The proof of eq. (A.2) is similar.

*End of proof of Lemma 1.*

We have also obtained a partial converse of Lemma 1:

*Lemma 2: There exists a correct classical protocol such that inequality (A.1) is saturated, and there exists a correct classical protocol such that inequality (A.2) is saturated. There also exists a correct classical protocol for which*

$$(1 - p_{0*})(1 - p_{*1}) = (1 - p_{1*})(1 - p_{*0}) = \frac{p_{\perp\perp}}{2} . \quad (\text{A.3})$$

*Proof of Lemma 2.* Let us consider the following protocol:

Round 1: Alice excludes one of the outcomes. That is she chooses that the outcome of the protocol will be either in  $\{0, 1\}$  (she has excluded  $\perp$ ),  $\{0, \perp\}$  (she has excluded 1)

or  $\{1, \perp\}$  (she has excluded 0). She tells her choice to Bob. If she is honest she chooses randomly among these three possibilities with a priori probabilities  $q_{01}$ ,  $q_{0\perp}$ ,  $q_{1\perp}$ .

Round 2: Bob chooses which of the remaining two outcomes is the result of the protocol. He tells Alice what is his choice. Thus for instance if Alice told him that the outcome was in  $\{0, 1\}$ , Bob can choose that the outcome is either 0 or 1, but not  $\perp$ . If he is honest he chooses randomly among the two remaining possibilities with probabilities  $q_{0|01}$ ,  $q_{1|01}$ ;  $q_{0|0\perp}$ ,  $q_{\perp|0\perp}$ ;  $q_{1|1\perp}$ ,  $q_{\perp|1\perp}$ .

It is easy to check that, if the parties are honest, the probabilities are:

$$\begin{aligned} p_{00} &= q_{0|01}q_{01} + q_{0|0\perp}q_{0\perp} \\ p_{11} &= q_{1|01}q_{01} + q_{1|1\perp}q_{1\perp} \\ p_{\perp\perp} &= q_{\perp|0\perp}q_{0\perp} + q_{\perp|1\perp}q_{1\perp} ; \end{aligned} \tag{A.4}$$

and that, if they are dishonest, the probabilities are:

$$\begin{aligned} p_{*0} &= \max\{q_{0|01}, q_{0|0\perp}\} \\ p_{*1} &= \max\{q_{1|01}, q_{1|1\perp}\} \\ p_{0*} &= q_{01} + q_{0\perp} \\ p_{1*} &= q_{01} + q_{1\perp} . \end{aligned} \tag{A.5}$$

If we choose the parameters such that  $q_{0\perp} = q_{1\perp}$ ,  $q_{0|01} = q_{1|01} = 1/2$ ,  $q_{0|0\perp} = q_{1|1\perp} \geq 1/2$ , then the protocol is correct and eq. (A.3) is verified.

And if we choose  $q_{0\perp} = 0$  then we have  $p_{\perp\perp} = q_{\perp|1\perp}q_{1\perp} = (1 - q_{1|1\perp})(1 - q_{01})$  and  $p_{0*} = q_{01}$ ,  $p_{*1} = q_{1|1\perp}$  thus saturating eq. (A.1). Note that by adjusting the remaining free parameter  $q_{0|01}$  one can make the protocol correct.

Similarly one can saturate inequality (A.2).

*End of proof of Lemma 2.*

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